

Keplerian periodogram for Doppler exoplanets detection: optimized computation and analytic significance thresholds

Roman V. Baluev*

*Central Astronomical Observatory at Pulkovo of Russian Academy of Sciences, Pulkovskoe shosse 65, St Petersburg 196140, Russia
Sobolev Astronomical Institute, St Petersburg State University, Universitetskij prospekt 28, Petrodvorets, St Petersburg 198504, Russia*

Accepted 2014 October 19. Received 2014 October 17; in original form 2014 June 17

ABSTRACT

We consider the so-called Keplerian periodogram, in which the putative detectable signal is modelled by a highly non-linear Keplerian radial velocity function, appearing in Doppler exoplanetary surveys. We demonstrate that for planets on high-eccentricity orbits the Keplerian periodogram is far more efficient than the classic Lomb-Scargle periodogram and even the multiharmonic periodograms, in which the periodic signal is approximated by a truncated Fourier series.

We provide new numerical algorithm for computation of the Keplerian periodogram. This algorithm adaptively increases the parameteric resolution where necessary, in order to uniformly cover all local optima of the Keplerian fit. Thanks to this improvement, the algorithm provides more smooth and reliable results with minimized computing demands.

We also derive a fast analytic approximation to the false alarm probability levels of the Keplerian periodogram. This approximation has the form $(Pz^{3/2} + Qz)W \exp(-z)$, where z is the observed periodogram maximum, W is proportional to the settled frequency range, and the coefficients P and Q depend on the maximum eccentricity to scan.

Key words: techniques: radial velocities - methods: data analysis - methods: statistical - surveys - stars: individual: HD80606

1 INTRODUCTION

So far, the Doppler radial-velocity (RV) monitoring is one of the most efficient exoplanets detection methods, both in the number of the planets discovered and in the amount of information obtained per an individual planet or planetary system.

The first exoplanet discovered by this method, 51 Pegasi *b*, induced a single and practically sinusoidal Doppler signal with an amplitude of approximately 60 m/s and a period of 4.2 d (Mayor & Queloz 1995). Thanks to the continuous growth both of the RV data amount and of the time base, we became able to detect less massive planets, orbiting their stars at larger distances. Additionally, now we are frequently dealing with much more complicated planetary systems generating multicomponent, severely non-linear, and remarkably non-sinusoidal Doppler signals.

Obviously, this progress necessitates the use of considerably more advanced data-analysis tools than those available in 1995. In this paper we consider the detection of exoplanets moving along orbits with a large eccentricity. The largest

of currently known exoplanetary orbital eccentricities are above 0.9, e.g. the well-investigated case of HD 80606 (e.g. Wittenmyer et al. 2009). According to Schneider (1995) Extrasolar Planets Encyclopaedia, the largest orbital eccentricity among all exoplanets detected by radial velocities belongs to HD 20782 with $e = 0.97$ (O’Toole et al. 2009a). In Solar System these eccentricities are typical for comets rather than planets. From the other side, according there are only three so extreme exoplanets that reveal $e > 0.9$, and the average orbital eccentricity is only about 0.2 (even after removal of the hot Jupiters subsample, in which the eccentricities are usually small or zero). Therefore, high-eccentricity exoplanets are not typical. Nevertheless, we still have a noticeable set of about 50 exoplanets with rather large eccentricity, $e > 0.6$. This corresponds to ~ 8 per cent of the exoplanets that were detected by radial velocities. This group of exoplanets is the one at which we focus our attention here.

An exoplanet moving along a highly eccentric orbit induces a drastically non-sinusoidal Keplerian Doppler signal that cannot be adequately modelled by a sinusoid. Consequently, the period search methods like the classic Lomb (1976)–Scargle (1982) periodogram, as well as any its extension that still models the putative signal with a plain

* E-mail: r.baluev@spbu.ru

sinusoid, would likely fail to detect such a planet, or at least they would not be very efficient. Planets on highly-eccentric orbits should be more efficiently detected by a periodogram in which the probe periodic signal is modelled by the Keplerian Doppler function with free (fittable) orbital parameters. Such a “Keplerian periodogram” was originally introduced by Cumming (2004). Later this periodogram proved rather useful in some complicated cases involving planets with large orbital eccentricities (e.g. O’Toole et al. 2007, 2009b). Thanks to a more accurate model of the non-sinusoidal planetary RV signal, the Keplerian periodogram allows a more efficient detection of high-eccentricity exoplanets and more reliable initial determination of their orbital parameters.

Since typical exoplanetary eccentricities are still not very large, the Keplerian periodogram should be treated as a specialized tool, rather than a mass-usage replace for more traditional period-search tools like e.g. the Lomb–Scargle periodogram. However, this does not remove the need of a special treatment for high-eccentricity exoplanets on their detection stage. Moreover, the high-eccentricity exoplanets ($e > 0.6$) are more difficult to detect, so their apparently small number can be due to an observational selection effect in some part. For example, Cumming (2010) argues that “there is good agreement that the detectability falls off for $e > 0.5 - 0.6$ ”. This only emphasizes the value of specialized detection tools designed to properly handle large eccentricities.

The Keplerian periodogram did not attain higher popularity due yet another reason. The Keplerian RV model depends on unknown parameters in a severely non-linear manner. This forces us to use some iterative non-linear fitting algorithms that increase the computation complexity dramatically. However, after the computation algorithm is implemented, the evaluation of an individual Keplerian periodogram is still a feasible task for modern CPUs. The more difficult issue is that until recently no useful method to calculate the significance thresholds for such periodograms was available. These thresholds are necessary to distinguish the real signal from noisy periodogram peaks. Monte Carlo simulation is no longer an option here, because it needs thousands of simulated Keplerian periodogram to be processed before we may have a good estimation of the necessary false alarm probability (FAP). The analytic computation of the periodogram FAP is a task that does not have an obvious solution even in the Lomb–Scargle case, whereas for Keplerian periodograms it is even more difficult.

Although Cumming (2004) gave some argumentation concerning the analytic or semi-analytic approximation of the FAP, after a close investigation, we find these conclusions unreliable and sometimes even mistaken, because they appear to implicitly neglect certain important non-linearity effects (to be discussed in more details below). The primary goal of this paper is to construct more strict approximations to the Keplerian periodogram FAP, involving a more careful treatment of the Keplerian non-linearity. We achieved this goal by means of the generalized Rice method, which is an approach of the modern probabilistic theory of extreme values of random processes and fields. Previously this method demonstrated a high efficiency in characterizing the significance levels of periodograms involving linear models (Baluev 2008), and recently it was adapted to periodograms that involve a general non-linear signal model (Baluev 2013c). Ba-

sically, now we just need to substitute the Keplerian model in the general formulae from (Baluev 2013c), although this task appeared technically hard.

The advantages of the Rice method are: (i) it is mathematically strict, (ii) it is very general, (iii) it yields an entirely analytic FAP estimations, eliminating the need of any Monte Carlo simulations, (iv) the final FAP estimations usually can be expressed by simple elementary formulae, (v) these approximations usually appear rather accurate, (vi) even if they are not very accurate they still serve as an upper limit on the FAP, guaranteeing that the actual false positives rate is at least limited by the desired level. Therefore, this approach remains unbeaten so far, although some promising fresh results were obtained by fitting the periodogram FAP with the extreme-value distributions (Süveges 2014).

The structure of the paper is as follows. First of all, in Sect. 2, we give a general overview systematizing the family of so-called “likelihood-ratio periodograms”, to which the Keplerian periodogram belongs as a special case. In Sect. 3 we provide the formal definition of the Keplerian periodogram in a bit more general formulation than Cumming (2004). In Sect. 4 we describe the application of the Rice method to the Keplerian periodogram and give the corresponding analytic FAP estimations. In Sect. 5 we describe an improved computing algorithm for the Keplerian periodogram, involving an optimized sampling of the space of Keplerian parameters. In Sect. 6 we demonstrate this algorithm using the system of HD 80606 as a rather complicated testcase. In Sect. 7 we provide an analytic comparison between the Keplerian and the sinusoidal models in view of their signal detection efficiency. In Sect. 8 we perform Monte Carlo simulations to verify the accuracy of the analytic FAP estimations of Sect. 4 and their applicability in practical situations.

2 OVERVIEW OF THE LIKELIHOOD-RATIO PERIODOGRAMS

In a large part, this work offers a yet another contribution to our series of papers devoted to the characterization of the periodograms significance levels (Baluev 2008, 2009b, 2013b,c,d). All these papers deal with the so-called likelihood-ratio periodograms that are based on the likelihood-ratio statistic comparing two rival models of the data: “the base model”, describing the underlying variation expected to be always present in the data, and “the alternative model”, expressed as the sum of the base model and of the putative periodic signal of a given functional form. The detailed mathematical definitions will follow in Sect. 3, and see also (Baluev 2014a).

In view of a large number of periodograms introduced in this work series so far, we provide a graphical scheme systematizing them based on two properties: the linearity of the associated maximum-likelihood fitting task and the complexity of the model used to approximate the probe periodic signal. This is shown in Fig. 1. We need to give a few more comments concerning this scheme:

(i) All special linear cases shown in Fig. 1 in the left column do not involve in their signal model anything more than sinusoids. This model is either a single sinusoid (e.g. Lomb–Scargle periodogram or its close relative, the floating-mean

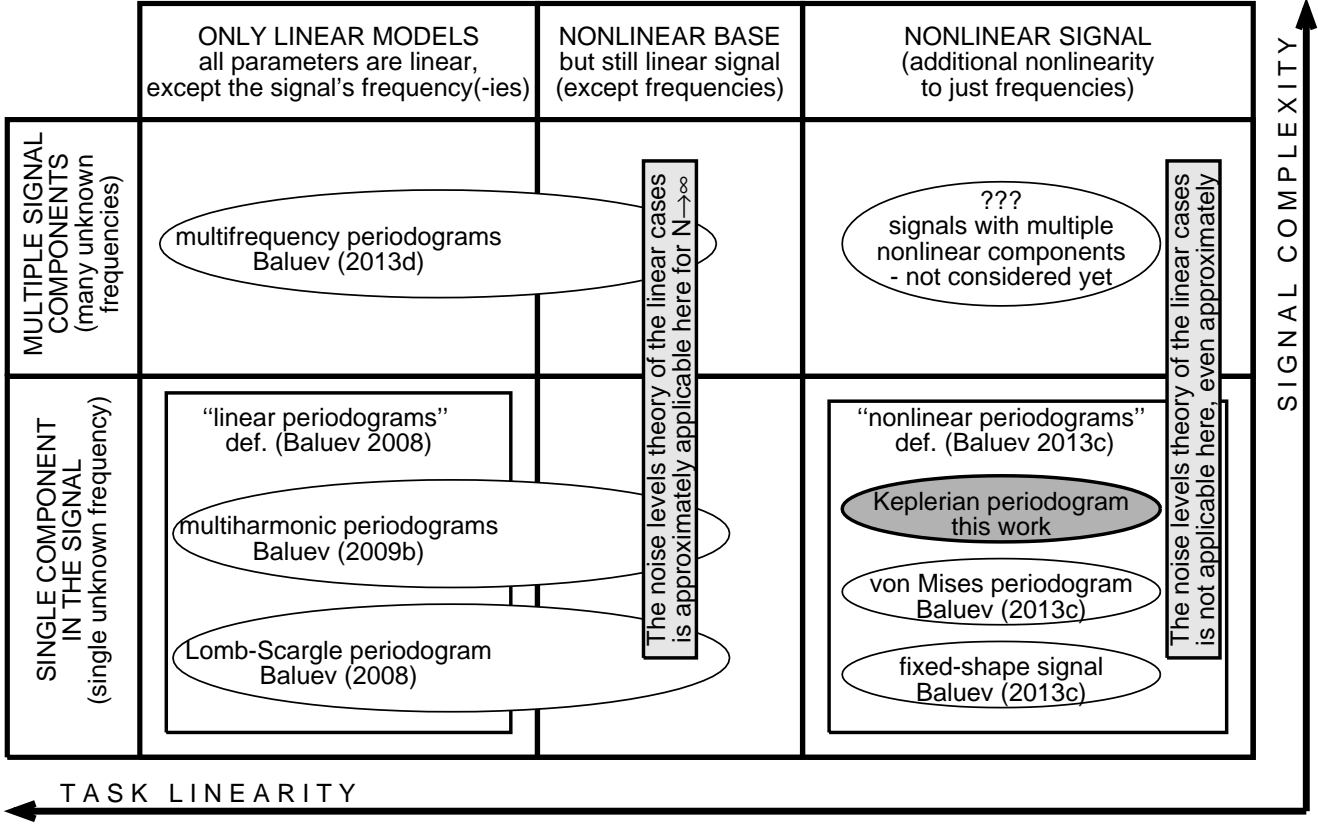


Figure 1. Schematic view of the periodograms based on the likelihood-ratio test, depending on the complexity of the periodic signal to fit and on the linearity of the fitting task.

periodogram by Ferraz-Mello 1981) or a sum of sinusoids (the multiharmonic and multifrequency periodograms). The difference between the multiharmonic and multifrequency periodograms is how the frequencies of the sinusoids are treated. In the first case they are binded with each other so that they form the sequence $f, 2f, 3f, \dots, nf$ with only a single basic frequency f to be determined. This approximates a non-sinusoidal signal by a partial sum of the Fourier series. In the second case all frequencies are free. In (Baluev 2008) a general theory of linear periodograms was given as well (with only a single fittable frequency). However, so far we never dealt with a periodic signal modelled by a function more complicated than a sum of sinusoids, but still linear.

(ii) There is a qualitative difference between the nonlinear periodograms in which the nonlinearity is caused by the base model (only) and by the model of the signal. The nonlinearity in the base model is usually “weak” in the sense that the significance levels can still be approximated by the relevant theory of linear periodograms (assuming the same signal model and a linearized or even just zero base model). To symbolically reflect this behaviour, in Fig. 1 we extend the circling of all the linear cases to the intermediate nonlinearity zone. The nonlinearity in the signal is essential and a computation of the significance levels for such periodograms requires a strict treatment of the nonlinear model, e.g. using the methods from (Baluev 2013a). See also a discussion in (Baluev 2014a).

(iii) Another qualitative difference is between a single-component (single unknown frequency) and multi-

component (many unknown frequencies) signals. To strictly handle the multicomponent variation it is not enough to e.g. extract the signal components one-by-one, just sequentially applying a periodogram with the relevant single-component signal model. To rigorously verify that we did not overestimate the number of the components in any of possible ways, we should consider the entire ensemble of the candidate components and separately test the significance of each their subsample. This requires to apply totally $2^n - 1$ periodograms with multicomponent signals, provided that n is the number of the candidate periodicities. See (Baluev 2013d) for a detailed discussion.

(iv) The most complicated case with nonlinear multicomponent signals is not investigated yet. What we aim to do in the present work is to consider the “Keplerian periodogram” in which the probe signal is modelled by a *single* Keplerian RV variation. This still does not provide an entirely rigorous basis for an analysis of multi-planet systems, because we will assume the classic simplified approach of extracting the planetary signatures one-by-one. The more rigorous treatment analogous to the multifrequency analysis (Baluev 2013d) is a considerably more complicated task in terms of both the associated theory and numerical computations. We leave this task of the scope of the paper.

(v) All periodograms shown in Fig. 1 belong to the general class of the so-called “recursive” periodograms, as named by Anglada-Escudé & Tuomi (2012). In our work we prefer to call them “residual” periodograms, contrary to the traditional periodograms of residuals. In the cases when

the base model is non-trivial and complicated, the residual periodograms may become considerably more efficient and should be preferred in practice. See an additional discussion in (Baluev 2014b, Sect. 5).

(vi) The references shown in Fig. 1 only reflect the works where the noise levels for a relevant periodogram were well characterized. Some of these periodograms were actually introduced in earlier papers belonging to other authors. The multiharmonic periodogram itself was considered by Schwarzenberg-Czerny (1996), while the multifrequency ones by Foster (1995), and the Keplerian periodogram was introduced by Cumming (2004).

In this paper we do not provide any comparison with the Bayesian model selection (e.g. Cumming 2004; Tuomi & Anglada-Escudé 2013), and in particular with the Bayesian Keplerian periodogram by Gregory (2007a,b). Although some comparison of this type was initially planned here, the resulting material appeared far more wide than the reasonable paper size limits would allow, so we plan to release these results in a separate work in the future.

3 KEPLERIAN PERIDOGRAMS

The RV variation induced by an unseen (planetary) satellite moving on a Keplerian orbit is given by the following formula:

$$\mu = K \{ \cos[\omega + v(\lambda - \omega, e)] + e \cos \omega \}, \quad (1)$$

with four input parameters: the signal semiamplitude K , the mean longitude λ , the eccentricity e , and the argument of the pericenter ω . The true anomaly $v(M, e)$ is a function of the mean anomaly $M = \lambda - \omega$ and of the eccentricity. Finally, the mean longitude can be represented as $\lambda = \lambda_0 + n(t - t_0)$, where λ_0 is the value of λ at some reference epoch t_0 , and n is the mean motion, which can be tied to the period $P = 2\pi/n$ or to the frequency $f = n/(2\pi)$. Therefore, the signal μ can be represented as a function of the time t and of five unknown parameters: the signal frequency f and the remaining parameters $\theta = \{K, \lambda_0, e, \omega\}$ (with a priori fixed t_0). We have separated the frequency f to comply with the notations of the work (Baluev 2013c) that we rely upon below.

In addition to the putative signal (1), the cumulative RV model should also contain some basic assumptions concerning the task — the base or null model. The base model should at least contain a constant term, and optionally the contributions from some other (already detected) planets similar to (1). The parameters of planetary contributions in the base model are approximately known, while the parameters of the signal are unknown (except for a small number of very wide limits, e.g. the admitted frequency range). The base model may be even more complicated, e.g. it can be a Newtonian N -body model in some cases. We define the base model as $\mu_{\mathcal{H}}(t, \theta_{\mathcal{H}})$, where $\theta_{\mathcal{H}}$ stores all its parameters (including e.g. orbital frequencies and other parameters of known planets).

Given the base model $\mu_{\mathcal{H}}(t, \theta_{\mathcal{H}})$, the alternative one $\mu_{\mathcal{K}}(t, \theta_{\mathcal{K}}) = \mu_{\mathcal{H}}(t, \theta_{\mathcal{H}}) + \mu(t, \theta, f)$ with $\theta_{\mathcal{K}} = \{\theta_{\mathcal{H}}, \theta, f\}$, and the input RV time series, we ask: how much the alternative model improves the fit of the data, in comparison with the base model fit?

Let us define the input time series as $\{t_i, x_i, \sigma_{i,\text{meas}}\}_{i=1..N}$ with t_i being the time of an i th observation, x_i being the actual RV measurement, and $\sigma_{i,\text{meas}}$ being its stated (probably incomplete) uncertainty. The measurements x_i incorporate the random errors ϵ_i , which are assumed independent and Gaussian. Now we can use several test statistics to compare $\mu_{\mathcal{H}}$ with $\mu_{\mathcal{K}}$. These tests differ by the adopted noise model.

The first group of tests is based on the classic least-square fitting. Let us define the goodness-of-fit function as

$$\chi_{\mathcal{H},\mathcal{K}}^2 = \langle (x - \mu_{\mathcal{H},\mathcal{K}})^2 \rangle, \quad (2)$$

where the operation $\langle \phi(t) \rangle$ is the weighted sum of $\phi(t_i)$ taken with weights $w_i = 1/\sigma_{i,\text{meas}}^2$ (see Baluev 2008 for the formal definition). We may fit the RV models involved by means of minimizing the function (2). In the definitions below we are not interested in the fitted values of the parameters, but we need the following χ^2 minima:

$$g_{\mathcal{H}} = \min_{\theta_{\mathcal{H}}} \chi_{\mathcal{H}}^2 \Big|_{\theta_{\mathcal{H}} \approx \theta_{\mathcal{H}}^0}, \quad g_{\mathcal{K}} = \min_{\theta_{\mathcal{H}}, \theta, f} \chi_{\mathcal{K}}^2 \Big|_{\theta_{\mathcal{H}} \approx \theta_{\mathcal{H}}^0}, \quad (3)$$

where $\theta_{\mathcal{H}}^0$ stands for the known initial (approximate) value of $\theta_{\mathcal{H}}$, and the notation $\theta_{\mathcal{H}} \approx \theta_{\mathcal{H}}^0$ indicates that the minimizations in (3) are performed locally over $\theta_{\mathcal{H}}$ (i.e., within the local maximum covering the initial guess). The minimization over θ is global. Based on the minimized χ^2 values, we may introduce the following tests:

$$z = (g_{\mathcal{H}} - g_{\mathcal{K}})/2, \quad z_3 = \frac{N_{\mathcal{K}}}{2} \ln \frac{g_{\mathcal{H}}}{g_{\mathcal{K}}}, \quad (4)$$

where $N_{\mathcal{K}} = N - d_{\mathcal{K}}$ with $d_{\mathcal{K}} = \dim \theta_{\mathcal{K}}$ (similarly we can define $N_{\mathcal{H}} = N - d_{\mathcal{H}}$ with $d_{\mathcal{H}} = \dim \theta_{\mathcal{H}}$). The test statistics (4) are direct generalizations of the linear periodograms $z(f)$ and $z_3(f)$ from (Baluev 2008). The statistic z is designed for the case when $\sigma_{i,\text{meas}}$ represent accurate values for the standard deviations of ϵ_i . This is rarely true. Instead we may use the statistic z_3 , which appears when adopting the classic noise model: $\mathbb{D}\epsilon_i = \kappa \sigma_{i,\text{meas}}^2$, where κ is an unknown scale factor (which is implicitly estimated from the data). This z_3 is proportional to the likelihood ratio statistic associated to the models $\mu_{\mathcal{H}}$ and $\mu_{\mathcal{K}}$. The proportionality factor is $N_{\mathcal{K}}/N$ and it was introduced mainly to make the test more conservative in the overfit case, when the dimensionality of the models is large (in comparison with N). The asymptotic behaviour of the statistic z_3 for $N \rightarrow \infty$ is the same as for the original likelihood ratio statistic.

However, the classic noise model is inappropriate for exoplanetary Doppler surveys. The noise variances should be better expressed as $\mathbb{D}\epsilon_i = \sigma_{\star}^2 + \sigma_{i,\text{meas}}^2$, where σ_{\star} is the RV jitter (Wright 2005). A simple fitting method handling this noise model, requires some a priori value of σ_{\star} , taken from empiric relations binding σ_{\star} to spectral activity indicators. Given the value of σ_{\star} , we may apply the least square approach above setting $w_i = 1/(\sigma_{\star}^2 + \sigma_{i,\text{meas}}^2)$. At present this is probably the most popular approach, since it introduces only minimum modifications to the least squares method and thus is easy to implement. However, as we have shown in (Baluev 2009a), the estimated values of σ_{\star} is usually very uncertain and also significantly depends on the spectrograph and even on the spectrum reduction algorithm used to obtain the actual RV measurements. In practice it is better to deal with a *fittable* noise model $\mathbb{D}\epsilon_i = \sigma_i^2(p) = p + \sigma_{i,\text{meas}}^2$,

where p is an additional free parameter. In this case we may use the maximum-likelihood method to estimate the joint vector of the parameters $\{p, \boldsymbol{\theta}_{\mathcal{H}, \mathcal{K}}\}$. For Gaussian noise, this likelihood function for the models $\mu_{\mathcal{H}}$ and $\mu_{\mathcal{K}}$ should look like

$$\ln \mathcal{L}_{\mathcal{H}, \mathcal{K}} = -\frac{1}{2} \sum_{i=1}^N \left\{ \ln \sigma_i^2(p) + \frac{[x_i - \mu_{\mathcal{H}, \mathcal{K}}(t, \boldsymbol{\theta}_{\mathcal{H}, \mathcal{K}})]^2}{\sigma_i^2(p)} \right\} + C, \quad (5)$$

where $C = N \ln \sqrt{2\pi}$ is a constant. In practice, we prefer to use a modified likelihood function from (Baluev 2009a):

$$\ln \tilde{\mathcal{L}}_{\mathcal{H}, \mathcal{K}} = -\frac{1}{2} \sum_{i=1}^N \left\{ \ln \sigma_i^2(p) + \frac{[x_i - \mu_{\mathcal{H}, \mathcal{K}}(t, \boldsymbol{\theta}_{\mathcal{H}, \mathcal{K}})]^2}{\gamma_{\mathcal{H}, \mathcal{K}} \sigma_i^2(p)} \right\} + C, \quad (6)$$

where $\gamma_{\mathcal{H}, \mathcal{K}} = 1 - d_{\mathcal{H}, \mathcal{K}}/N$. This modification involves a preventive bias reduction for the fitted value of p : the corrector γ basically increases the residuals, which are always systematically smaller than real errors ϵ_i .

By analogy with (3) we obtain the relevant likelihood function maxima

$$\tilde{l}_{\mathcal{H}} = \ln \max_{p, \boldsymbol{\theta}_{\mathcal{H}}} \tilde{\mathcal{L}}_{\mathcal{H}} \Big|_{\substack{p \approx p_0^0 \\ \boldsymbol{\theta}_{\mathcal{H}} \approx \boldsymbol{\theta}_{\mathcal{H}}^0}}, \quad \tilde{l}_{\mathcal{K}} = \ln \max_{p, \boldsymbol{\theta}_{\mathcal{H}}, \boldsymbol{\theta}, f} \tilde{\mathcal{L}}_{\mathcal{K}} \Big|_{\substack{p \approx p_0^0 \\ \boldsymbol{\theta}_{\mathcal{H}} \approx \boldsymbol{\theta}_{\mathcal{H}}^0}}, \quad (7)$$

and the associated modified likelihood-ratio statistic from (Baluev 2009a):

$$\tilde{Z} = \frac{N_{\mathcal{K}}}{N} (\tilde{l}_{\mathcal{K}} - \tilde{l}_{\mathcal{H}}) + \frac{N_{\mathcal{K}}}{2} \ln \frac{N_{\mathcal{H}}}{N_{\mathcal{K}}}. \quad (8)$$

The offset and normalization of (8) was chosen so that for the classic noise model $\tilde{Z} = z_3$, and for $N \rightarrow \infty$ the asymptotic behaviour of \tilde{Z} is the same as for the conventional likelihood-ratio statistic (i.e. for $Z = \max \ln \mathcal{L}_{\mathcal{K}} - \max \ln \mathcal{L}_{\mathcal{H}}$).

Technically, the Keplerian periodogram may be based on any of the three statistics, z , z_3 , or \tilde{Z} , although \tilde{Z} is the one preferable in practice. The definitions (4) and (8) involve the maximizations over all free parameters, including $\boldsymbol{\theta}$ and f . These statistics represent just some scalar values that correspond to the maxima of the relevant Keplerian periodograms that we still need to define. Traditionally, the periodograms are represented as functions of the signal frequency (or period). Therefore, given some probe frequency, we should compute the quantities similar to (4) and (8), but performing the maximizations in (3) and (7) fixing the frequency at the selected value. The resulting functions of the frequency are our Keplerian periodograms: $z(f)$, $z_3(f)$, and $\tilde{Z}(f)$. Below we will rarely use these periodograms themselves, mainly dealing with their maxima over f . Therefore, to prevent further increase in the number of the notations we will distinguish the Keplerian periodograms (e.g. $z(f)$) from their maxima (e.g. z) only by the dependence on f , which in the first case will be always shown explicitly.

4 ANALYTIC STATISTICAL THRESHOLDS FOR THE KEPLERIAN PERIDOGRAM

The parameters f , K , and λ_0 of the Keplerian model (1) are present in the simple sinusoidal model too. Thus, in comparison with the Lomb-Scargle periodogram, the Keplerian periodogram adds two more degrees of freedom with the parameters e and ω . Simultaneously, it adds more non-linearity to the task. The only obvious linear parameter of (1) is the

semi-amplitude K . There are ways to rewrite this model so that *two* linear parameters appear (Zechmeister & Kürster 2009) instead of only a single K . However, the remaining two parameters and the frequency are still non-linear.

It is well known that the statistical significance thresholds for a periodogram are closely tied to the distributions of the test statistic involved in the periodogram definition. To compute the FAP for a detected signal we must assess the distribution of this statistic under an assumption that the data contain nothing but the underlying variation (given by the model $\mu_{\mathcal{H}}$) and noise. Cumming (2004) have already considered this task for the Keplerian periodogram $z(f)$ and some its close relatives. In particular, he advocated that for $N \rightarrow \infty$ the value of the Keplerian periodogram $z(f)$ (with a fixed f) should asymptotically obey the χ^2 distribution with 4 degrees of freedom (because we have 4 free parameters of the model, except the frequency). But after a preliminary investigation of the task, we find that this conclusion is likely mistaken.

The asymptotic χ^2 approximation to the distribution of a test statistic like z would appear only if the models $\mu_{\mathcal{H}}$ and $\mu_{\mathcal{K}}$ were both linearizable in the point where we want to compute the distribution (i.e. at the point where $\mu \equiv 0$). This is not true for the Keplerian model (1): it cannot be linearized at $K = 0$ without degeneracies. This issue is discussed in more details in (Baluev 2013c), and it originates in the non-identifiability of the Keplerian parameters at $K = 0$. We cannot construct any Taylor decomposition of (1) at $K = 0$ that would be functional for all possible values of other parameters. In other words, the Keplerian parameters are essentially non-linear here and are similar to the frequency in this concern. By fixing only the frequency we cannot eliminate or reduce the Keplerian non-linearity, even in any asymptotic or approximate sense. Therefore, the χ^2 distribution is not a good approximation here, regardless of whether we have the frequency fixed or free. Therefore, we do not rely on the results by (Cumming 2004) in what concerns the significance estimations.

Cumming (2004) have done some Monte Carlo simulations that apparently confirmed his conclusions about the distributions of the Keplerian periodogram. However, it seems that his computation of the Keplerian periodogram suffers from undersampling effects that are discussed below in Sect. 5. This makes his simulation results unreliable. This is basically the case in which two distorting effects act in opposite directions and thus largely compensate and hide each other.

Analytic approximations to the significance levels of the Keplerian periodogram can be derived using the method introduced in (Baluev 2013c). This method was developed for periodograms based on the chi-square z statistic with an arbitrary non-linear model of the periodic signal μ , but assuming that $\mu_{\mathcal{H}}$ depends on $\boldsymbol{\theta}_{\mathcal{H}}$ in a linear manner. This theory gives the FAP estimations in the form:

$$\text{FAP} \lesssim M(z), \quad (9)$$

with $M(z)$ depending on the structure of the signal model and on the parametric domain.

For example, in the simplest case the signal is given by the sinusoid. Its harmonic coefficients are unbounded, while the frequency is limited to a segment:

$$0 < f < f_{\max}. \quad (10)$$

This family of “linear periodograms”, including the classic Lomb-Scargle periodogram, was considered in (Baluev 2008), where the following was found:

$$\begin{aligned} \text{FAP} &\lesssim M(z) \approx W e^{-z} \sqrt{z}, \\ W &= f_{\max} T_{\text{eff}}, \quad T_{\text{eff}} = \sqrt{4\pi (\bar{t}^2 - \bar{t}^2)}. \end{aligned} \quad (11)$$

Here the quantities \bar{t}^k represent the weighted averages of observation times t_i (taken with the weights appearing in the chi-square function).

In fact, the work (Baluev 2013c) develops an extension of the same method to a generalized “non-linear periodogram”, in which the signal is modelled by an almost arbitrary non-linear (and non-sinusoidal) periodic function. Now we deal with a specialized case. The signal is given by the non-linear model (1), and its parametric domain is defined as:

$$\begin{aligned} K > 0, \quad 0 < \lambda < 2\pi, \quad 0 < f < f_{\max}, \\ 0 < e < e_{\max}, \quad 0 < \omega < 2\pi. \end{aligned} \quad (12)$$

Here we set upper limits on the frequency and on the eccentricity. The need to set finite frequency limits is not surprising: similar frequency limits are used for the Lomb-Scargle periodogram. The eccentricity limit is new. Below it is demonstrated that such a limit is necessary due to several reasons, including the singular behaviour of (1) when $e \rightarrow 1$. Note that the domain (12) does not contain pairs of duplicate Keplerian signals, which will be important for the correctness of the resulting FAP estimation.

The auxiliary parameter vectors used in (Baluev 2013c) now look like:

$$\boldsymbol{\theta} = \{K, \lambda_0, e, \omega\}, \quad \boldsymbol{\xi} = \{\lambda_0, e, \omega\}, \quad \boldsymbol{\nu} = \{e, \omega\}. \quad (13)$$

For further convenience we also define $\boldsymbol{\xi}' = \{f, \lambda_0, e, \omega\}$. Also, we need to extract the periodic shape function of the Keplerian variation (1), i.e. the part of μ that does not depend of K :

$$\begin{aligned} h(t, \boldsymbol{\xi}') &= g(\lambda_0 + 2\pi f(t - t_0), \boldsymbol{\nu}), \\ g(\lambda, \boldsymbol{\nu}) &= \cos[\omega + \nu(\lambda - \omega, e)] + e \cos \omega. \end{aligned} \quad (14)$$

After that we should define a properly normalized model $\psi(t, \boldsymbol{\xi}')$ such that

$$\langle \psi \mu \mathcal{H} \rangle \equiv 0, \quad \langle \psi^2 \rangle \equiv 1. \quad (15)$$

The general formula for ψ is given in (Baluev 2013c). After that, we need to calculate the following matrices that describe the local metric of the likelihood function:

$$\mathbf{G}_f = \left\langle \frac{\partial \psi}{\partial \boldsymbol{\xi}} \otimes \frac{\partial \psi}{\partial \boldsymbol{\xi}} \right\rangle, \quad \mathbf{G} = \left\langle \frac{\partial \psi}{\partial \boldsymbol{\xi}'} \otimes \frac{\partial \psi}{\partial \boldsymbol{\xi}'} \right\rangle, \quad (16)$$

corresponding to the cases of fixed f or free f , respectively. Clearly, \mathbf{G}_f is a submatrix of \mathbf{G} , since $\boldsymbol{\xi}$ is a subvector of $\boldsymbol{\xi}'$.

At first, let us apply the approximate approach (Baluev 2013c, sect. 4.1), based on the assumption of “uniform phase coverage”. In this approach the matrix \mathbf{G} is approximated by

$$\mathbf{G} \approx \begin{pmatrix} 4\pi^2 \bar{t}^2 q & 2\pi \bar{t} q & 2\pi \bar{t} \mathbf{v}^T \\ 2\pi \bar{t} q & q & \mathbf{v}^T \\ 2\pi \bar{t} \mathbf{v} & \mathbf{v} & \mathbf{V} \end{pmatrix}, \quad (17)$$

where the quantity q , the vector \mathbf{v} , and the matrix \mathbf{V} are expressed as

$$\begin{aligned} q &= \frac{\overline{g_\lambda'^2}}{g^2}, \quad v_i = \frac{\overline{g'_\lambda g'_{\nu_i}}}{g^2} - \frac{\overline{g g'_\lambda} \overline{g g'_{\nu_i}}}{g^2}, \\ V_{ij} &= \frac{\overline{g'_{\nu_i} g'_{\nu_j}}}{g^2} - \frac{\overline{g g'_{\nu_i}} \overline{g g'_{\nu_j}}}{g^2}, \quad R_{ij} = V_{ij} - \frac{v_i v_j}{q}. \end{aligned} \quad (18)$$

In the last formulae, the function g should be substituted from (14), the over-lines denote the integral averaging over the periodic argument λ . The function g should necessarily satisfy here the prerequisite condition $\bar{g} = 0$, which for our Keplerian model is already fulfilled. Note that the continuous averaging operation used in (18) is thus different from the discrete averaging \bar{t}^k from (11) and (17).

The most hard part of the work is the computation of (18). The second term in the expression for v_i vanishes, because the integral of $(g g'_\lambda)$ over a single period of λ is obviously zero (this useful property was actually missed in Baluev 2013c, as it is valid for an arbitrary smooth and λ -periodic g). But the other terms appearing in (18) are non-trivial. The schematic plan of the computation contains two steps:

(i) Derive the necessary derivatives of the Keplerian shape function g . In fact, these derivatives are already available in various literature (Pál 2010; Wright & Howard 2009).

(ii) Compute the necessary averages of the combinations appearing in (18) over λ . The derivatives of g involve the functions of the type $r^n \cos kv$ and $r^n \sin kv$, where r is the Keplerian radius-vector. Averages of such expressions can be found in handbooks on the Keplerian motion (e.g. Kholshevnikov & Titov 2007).

Regardless of principal feasibility, the manual computation of (18) is an extremely difficult task. We have undertaken a few such attempts, and none of them was successful due to mistakes appearing in the process. We eventually decided to use the MAPLE computer algebra system to obtain more reliable expressions for (18). The details of these computations are given in the MAPLE worksheet attached as the online supplement to the article. After some polishing of the MAPLE results, we obtain the following:

$$\begin{aligned} q &= \frac{(1 + \beta^2)^3}{(1 - \beta^2)^5} \frac{B(\beta, \omega)}{A(\beta, \omega)}, \\ v_\omega &= \left(\frac{1 + \beta^2}{1 - \beta^2} \right)^2 \frac{1}{A(\beta, \omega)}, \quad v_\beta = \frac{\beta(1 + \beta^2)}{(1 - \beta^2)^3} \frac{\sin 2\omega}{A(\beta, \omega)}, \\ V_{\omega\omega} &= \frac{1 - \beta^4}{A^2(\beta, \omega)}, \quad V_{\omega\beta} = \frac{\beta \sin 2\omega}{A^2(\beta, \omega)}, \\ V_{\beta\beta} &= \frac{C(\beta, \omega)}{(1 - \beta^4) A^2(\beta, \omega)}, \\ R_{\omega\omega} &= \frac{\beta^2(1 - \beta^4)(4 + \beta^2)}{A^2(\beta, \omega) B(\beta, \omega)}, \quad R_{\omega\beta} = \frac{\beta^3(4 + \beta^2) \sin 2\omega}{A^2(\beta, \omega) B(\beta, \omega)}, \\ R_{\beta\beta} &= \frac{(4 + \beta^2)(A(\beta, \omega) B(\beta, \omega) + \beta^4 \sin^2 2\omega)}{(1 - \beta^4) A^2(\beta, \omega) B(\beta, \omega)}, \\ \det \mathbf{R} &= \frac{\beta^2(4 + \beta^2)^2}{A^3(\beta, \omega) B(\beta, \omega)}, \end{aligned} \quad (19)$$

where

$$\begin{aligned} A &= 1 - \beta^2 \cos 2\omega, \quad B = A + \beta^2(4 + \beta^2), \\ C &= 4 + 5\beta^2 + \beta^4 - \beta^2(2 + \beta^2) \cos^2 \omega - 4\beta^4 \cos^4 \omega, \\ \beta &= \frac{e}{1 + \eta}, \quad \eta = \sqrt{1 - e^2}. \end{aligned} \quad (20)$$

Here we have replaced the eccentricity e by the new parameter β , because these formulae appear more seizable in terms of β . This might be suspected e.g. from the formulae of the integrals of $r^n \cos kv$ in (Kholshchikov & Titov 2007).

Substituting (19) to (17) and then to the suitable expressions for $M(z)$ from (Baluev 2013c), we obtain the FAP approximations of the following form:

$$\begin{aligned} \text{FAP}(z) &\lesssim M(z) = \exp(-z) \left[2zX_f(e_{\max}) + Y_f(e_{\max})\sqrt{\pi z} + \mathcal{O}(z^0) \right] \quad (\text{fixed } f), \\ \text{FAP}(z) &\lesssim M(z) = W \exp(-z) \sqrt{z} \left[2zX(e_{\max}) + Y(e_{\max})\sqrt{\pi z} + \mathcal{O}(z^0) \right] \quad (\text{free } f). \end{aligned} \quad (21)$$

The functions X and Y are expressed as

$$\begin{aligned} X_f(e_{\max}) &= \int_0^{\beta(e_{\max})} d\beta \int_0^{2\pi} \sqrt{q \det \mathbf{R}} \frac{d\omega}{2\pi}, \\ X(e_{\max}) &= \int_0^{\beta(e_{\max})} d\beta \int_0^{2\pi} q \sqrt{\det \mathbf{R}} \frac{d\omega}{2\pi}, \\ Y_f(e_{\max}) &= \int_0^{2\pi} \sqrt{q R_{\omega\omega}} \frac{d\omega}{2\pi} \Bigg|_{\beta=\beta(e_{\max})}, \\ Y(e_{\max}) &= \int_0^{2\pi} q \sqrt{R_{\omega\omega}} \frac{d\omega}{2\pi} \Bigg|_{\beta=\beta(e_{\max})}. \end{aligned} \quad (22)$$

In (21), the terms containing X reflect the expected number of local maxima of the likelihood function in the entire domain (12), while the terms with Y reflect the number of local maxima on the boundary of (12) at $e = e_{\max}$. The terms corresponding to the boundary $f = f_{\max}$ are negligible, because they do not contain the large factor W .

The FAP approximations (21) look rather similar to those for the von Mises periodogram discussed in (Baluev 2013c), with the eccentricity being an analogue of the localization parameter. And similarly to the von Mises periodogram, it appears that the coefficients X and Y tend to infinity if e is unbounded, so we must limit the eccentricity by some $e_{\max} < 1$ to have meaningful results. This e_{\max} should be selected a priori, like f_{\max} .

The integrals (22) are not elementary, except for X_f , for which we obtain (again with MAPLE):

$$X_f(e_{\max}) = \frac{\beta^2(24 - 21\beta^2 + 7\beta^4)}{12(1 - \beta^2)^3} \Bigg|_{\beta=\beta(e_{\max})}. \quad (23)$$

Most of the other integrals can be represented through pretty unpleasant combinations of elliptic integrals. Moreover, for $X(e)$ MAPLE obtains an indefinite result due to some tricky degeneracy appearing in the process. Perhaps this integral may involve something more complicated than

even the elliptic functions. The details are given in the attached MAPLE worksheet.

In any case, these accurate expressions are difficult for practical use, and we therefore tried to fit (22) numerically using some more simple formulae. After a few experiments, the following semi-empiric expressions were constructed:

$$\begin{aligned} X_f(e) &\simeq 0.5\epsilon^2 + 0.1042\epsilon^3 - 0.0914\epsilon^{2.44}, \\ X(e) &\simeq 0.5\epsilon^2 + 0.0350\epsilon^6 + 0.3334\epsilon^{3.86} + 0.0774\epsilon^{5.03}, \\ Y_f(e) &\simeq \epsilon + 0.5033\epsilon^3 + 0.2585\epsilon^{2.44}, \\ Y(e) &\simeq \epsilon + 0.3125\epsilon^6 + 2.3725\epsilon^{3.05} + 0.9868\epsilon^{4.86}, \\ \epsilon &= \frac{e}{\eta} = \frac{2\beta}{1 - \beta^2}. \end{aligned} \quad (24)$$

These approximations preserve the asymptotic behaviour of (22) for $e \rightarrow 0$ and for $e \rightarrow 1$, and their relative errors are below 1 per cent for X and Y and below 5 per cent for X_f and Y_f . It follows that for $e \rightarrow 1$ the FAP increases as either η^{-6} (for the free f) or η^{-3} (for the fixed f).

As we can see, although the procedure of computation was very hard in its internals, the final approximations (21) and (24) are not that complicated, and even became elementary. Also, one may note that the FAP of the fixed-frequency case clearly does not match the χ^2 distribution with 4 degrees of freedom. This χ^2 distribution would imply $\text{FAP}(z) = (z + 1) \exp(-z)$, and this is more or less close to the first formula of (21) only when $X_f = 0.5$, achieved for $e_{\max} \approx 0.7$.

The comparison of these theoretic results with the results of Monte Carlo simulations will be given in Sect. 8 below.

5 KEPLERIAN PERIODOGRAM COMPUTATION

Clearly, fitting the data with the non-linear model (1) is more complicated than fitting e.g. the sinusoidal model. It follows from (Cumming 2004; Zechmeister & Kürster 2009) that the likelihood function often has multiple maxima in the Keplerian case, and these maxima concentrate at high eccentricities. When computing the Keplerian periodogram, it is important to process *each* of these local maxima. Missing even a single such local maxima may result in underestimated likelihood-ratio statistic, and hence in a decreased detection power. This may also generate discontinuities in the graph of the Keplerian periodogram, occurring when the global likelihood maximum moves to a missed local one. Such discontinuities can be noticed in the plots by Zechmeister & Kürster (2009), and this indicates that the Keplerian parametric space might be undersampled in that work. It seems that Zechmeister & Kürster (2009) used some regular and uniform grid of the parameters for maximization, although they recognize that a non-uniform grid with the density increasing with e would be preferred. Cumming (2004) were “trying several initial starting values for the phases and eccentricity”. This likely means that the parametric space was undersampled too, as only “several” initial conditions is often too small in this task.

The algorithm that we propose here tries to achieve the best performance without sacrificing the safety of the final

result (that is, disallowing to loose local maxima). It is based on the following principles:

(i) Like Cumming (2004), we use the non-linear Levenberg-Marquardt optimization, subsequently trying starting conditions from a relatively rarified set (roughly one or a few points per each local maximum of the likelihood function). The plain grid-scanning approach used by Zechmeister & Kürster (2009) would require a much more dense grid, since they needed to also sample each local maximum at a high enough density.

(ii) In order to not miss any local maximum and avoid undersampling, we pay more attention to the construction of the grid of starting conditions. This grid is not uniform: its density adaptively increases together with the expected density of the local maxima.

(iii) Our algorithm does not currently use the main idea of the method by Zechmeister & Kürster (2009) to extract two linear parameters in the model (1) and treat them separately from the remaining non-linear ones. Neither we use the similar approach proposed by Wright & Howard (2009) for RV curves fitting.

Now our task is to construct an optimal multi-dimensional grid that would be rarified as much as possible, still disallowing any likelihood maxima to evade between the grid nodes. From (Zechmeister & Kürster 2009) and from Sect. 4 above we may conclude that local maxima of the likelihood function concentrate at high eccentricities, and this is confirmed by the formulae (21,24). The reason for such behaviour is that for large e the Keplerian signal (1) is almost constant most of the time, except for a short periastron passage events. The characteristic time spend by the planet near its orbital pericenter is inversely proportional to the pericentric angular velocity of the planet. From the second Kepler's law the planetary angular velocity can be determined as $\dot{\nu} = f(a/r)^2\eta$. This turns into $f\eta/(1-e)^2$ in the pericenter. Therefore, the planet spends near the pericenter roughly $(1-e)^2/\eta = \eta^3/(1+e)^2$ fraction of each its orbital period. This time decreases for large e as $\sim \eta^3$. To adequately trace such short spikes in the Keplerian radial velocity, some of the Keplerian parameters have to be sampled at an increasingly high density, when e tends to unity.

The optimal grid can be constructed using the results of Sect. 4. The integral for X in (22) is proportional to the expected number of the local maxima of the likelihood found inside the integration domain, while the integrand is proportional to the local density of these maxima. Therefore, the ideally optimal grid of Keplerian parameters should comply with the following probability density function (PDF):

$$p_{f\lambda\beta\omega} \propto \sqrt{\det \mathbf{G}} = q\sqrt{\det \mathbf{R}} = \beta(4+\beta^2) \frac{(1+\beta^2)^3}{(1-\beta^2)^5} \sqrt{\frac{B(\beta, \omega)}{A^5(\beta, \omega)}} \quad (25)$$

It is clear that:

$$p_{f\lambda e\omega} = p_f p_\lambda p_{e\omega}, \quad p_f = \frac{1}{f_{\max}}, \quad p_\lambda = \frac{1}{2\pi},$$

$$p_{e\omega} = \frac{q\sqrt{\det \mathbf{R}}}{2\pi X(e_{\max})} \frac{d\beta}{de}, \quad e \leq e_{\max}, \quad (26)$$

implying that f and λ_0 are uniformly-distributed and independent from each other and from e and ω , while e and ω are mutually correlated. It follows that the marginal PDF and

cumulative distribution function (CDF) of the eccentricity in the grid are given by

$$p_e = \frac{X'(e)}{X(e_{\max})}, \quad P_e = \frac{X(e)}{X(e_{\max})}, \quad e \leq e_{\max}, \quad (27)$$

and the biparametric PDF of e and ω can be expressed as

$$p_{e\omega} = p_e p_{\omega|e}, \quad p_{\omega|e} \propto \sqrt{\frac{B(\beta(e), \omega)}{A^5(\beta(e), \omega)}}. \quad (28)$$

We must emphasize that this approach necessarily requires a *random* rather than a regular grid. Even if λ_0 and f are independent from other parameters, we cannot just select two independent grids for λ_0 and f , some grid for e and ω , and form the resulting multidimensional grid as a Cartesian product of these three. In such a case the values of λ_0 and f would attain the same values for all e and ω , but this may lead to lost local maxima. To achieve an optimal grid, the values of λ_0 and f should be sampled anew for each new pair (e, ω) . This in fact implicitly simulates an increasingly more dense distribution of the sampled values of λ_0 and f when e grows: in a vicinity of a larger e the number of grid nodes is considerably larger, implying a smaller λ_0 - and f -separation between neighbouring nodes. In fact, some increasing of the phase and frequency resolution for larger e is a mandatory property of the required grid.

However, a disadvantage of a random grid is that it makes the computation results random. There is no guarantee that random nodes do not accidentally avoid some regions of the parametric space, potentially leading to lost local maxima. To suppress this effect we may increase the number of the generated nodes, but this makes the grid oversampled in other regions, slowing the periodogram computation down.

We therefore still need to have a regular grid, but this requires to correctly simulate the increase of the density in λ_0 and f . Note that for $e \rightarrow 1$ we have $p_e \sim \eta^{-8}$, which is an excessively quick growth. The local density of the likelihood function peaks along an e -isoline (fixing all parameters but e) can be computed by restricting the matrix \mathbf{G} to its single element $V_{\beta\beta}$, given in (19). The necessary density (for β) is proportional to $\sqrt{R_{\beta\beta}}$. Mapping it from β to e , we may obtain that the reasonable grid density along e -isolines should scale as $\sim \eta^{-5/2}$ (the pessimistic case with ω near 0 or π) or as $\sim \eta^{-3/2}$ (the optimistic case with ω near $\pm\pi/2$). We adopt the average rate of $\nu_e(e) \sim \eta^{-2}$. Since the original growth rate was $\sim \eta^8$, for each grid layer with fixed e we still have about $\sim \eta^6$ values of λ_0 and f to distribute uniformly and independently. The reasonable local density of the grid nodes along an ω -isoline is given by $\sqrt{R_{\omega\omega}}$, and their total number is proportional to $\int_0^{2\pi} \sqrt{R_{\omega\omega}} d\omega = 2\pi$. Expectedly, the total number of the grid nodes along an ω -isoline should remain constant for all e .

The most obvious way is to generate $\sim \eta^{-3}$ grid nodes per each λ_0 - and f -isoline (for a given e). This agrees with the expected peaks number along the isolines obtained after restriction of \mathbf{G} to the corresponding diagonal element and integration of its square root over the selected parameter. Both for the λ_0 - and f -isolines, we have the number of peaks proportional to \sqrt{q} , corresponding to the growth rate of $\sim \eta^{-5/2}$ (optimistic cases $\omega = 0$ or π) to $\sim \eta^{-3}$ (pessimistic

cases $\omega = \pm\pi/2$). Our grid corresponds to the pessimistic case here. The total number of the grid nodes behaves as $\int_0^{e_{\max}} \nu_e \eta^{-6} de \sim \eta_{\min}^{-6}$ for $e \rightarrow 1$, which is exactly the growth rate of the average number of peaks given by $X(e_{\max})$.

In fact, to strictly ensure that no potential peak is missed, we should distribute the grid nodes along all parametric isolines according to the corresponding pessimistic cases. We violated this rule in the case of the eccentricity isoline, for which the pessimistic density growth rate should be $\eta^{-2.5}$ instead of the adopted η^{-2} . Otherwise we would have the total number of the grid nodes growing as $\eta_{\min}^{-6.5}$, which is slightly larger than the total number of the peaks given by $X(e_{\max})$. Such grid would be slightly oversampled in comparison with an ideal (optimal) one. This oversampling appeared because the geometry of the peaks may be distorted by the off-diagonal elements of \mathbf{G} : a single peak may become elongated and inclined, spanning across several grid layers, being counted in each. To equate the number of grid nodes with the number of the expected peaks, we neglected the multiplier of $1/\sqrt{\eta}$ in the eccentricity density function. Even for $e = 0.9$ we have $\sqrt{\eta} \approx 0.66$, which does not differ much from unity. In practice the need to handle $e > 0.9$ emerges only in very rare extreme cases.

Also, we do not take into account the non-uniform distribution of ω . It follows from the above discussion that the sampled values of ω should concentrate somewhat to the values of 0 and π . However, the required concentration looks rather weak. Assuming that the grid density should vary proportionally to $\sqrt{V_{\omega}}$ (the density of the peaks found on the given ω -isoline), we obtain that the cumulative number of such nodes in a variable range $[0, \omega]$ should be proportional to $\arctan(\tan \omega / \sqrt{\eta})$. This means that the necessary grid nodes can be generated as $\omega = \arctan(\sqrt{\eta} \tan \alpha)$, where α is an auxiliary uniformly distributed angle. Due to the same rather mild factor of $\sqrt{\eta}$, for $e < 0.9$ this distribution does not differ very much from the uniform one.

Summarizing, our parametric grid can be constructed using the following instructions:

- (i) Sample the eccentricity e according to the formula $e_k = \sqrt{1 - \exp(-(2k+1)h_e)}$, where h_e controls the eccentricity resolution. This discrete distribution corresponds to the eccentricity density function of e/η^2 (uniform in $\ln \eta$), satisfying the $e \rightarrow 1$ asymptotic of $\sim \eta^{-2}$ requested above.
- (ii) For each sampled value of e , construct the grid in f with a step of $h_f \eta^3 / T$, where h_f is a control parameter.
- (iii) For each sampled value of e , construct the grid in λ_0 with a step of $h_\lambda \eta^3$, where h_λ is another control parameter.
- (iv) For each sampled value of e , construct the grid in ω with some step h_ω .

This grid has four control parameters h_e, h_f, h_λ , and h_ω that allow to control the absolute resolution of the parameters. We recommend the following values: $h_e = 1/6$, $h_f = 1/2$, $h_\lambda = \pi$, and $h_\omega = \pi/2$. Note that the eccentricity should be sampled first, and the remaining Keplerian parameters f , λ_0 , and ω can be sampled independently from each other, but depending on e .

It is rather obvious that the mean longitude λ_0 , basically the phase of the signal, should be sampled at higher density for large e : to locate the position of a narrow peak in the high-eccentricity Keplerian curve (marking the periastron

passage time) we should try to fit many template Keplerian curves, each shifted by the width of the peak. However, the similar property of the frequency might be surprising, although it might be suspected e.g. from a similar property of the multiharmonic periodograms that require finer frequency resolution with a larger order (Baluev 2009b). Consider that the model (1) approximates the true signal in the middle of the observation segment, but the model frequency is shifted from the true value by some Δf . Then near the ends of the time segment the deviation between the phases of the model and of the true signal would be $\pm \Delta f T / 2$. This quantity should be at least the as the step $\Delta \lambda_0$, because otherwise the periastron passages near the ends of the time segment might displace too much, leading to an inadequate model.

After the grid is ready, we may run the Levenberg-Marquardt fitter per each grid point, taking it as a starting approximation. The maximum likelihood attained over the grid represents the desired value of the Keplerian periodogram. The associated best fitting Keplerian parameters represent some useful by-product data.

A prototype algorithm of the Keplerian periodogram computation, based on the statistic \tilde{Z} from (8), was included in the PlanetPack software (Baluev 2013a) as of version 1.6. In the forthcoming version 2.0 this algorithm will be released in the improved form, including new optimized grid of Keplerian parameters described above.

6 KEPLERIAN PERIDOGRAM IN ACTION

As a good test suite for the Keplerian periodogram we consider the public RV data from ELODIE (Naef et al. 2001), Keck (Butler et al. 2006), and HET (Wittenmyer et al. 2007)¹ for the famous star HD 80606, hosting a unique planet that moves along an extremely elongated orbit ($e = 0.93$).

First of all, we tried to process these data using more traditional sinusoidal model of the signal. As expected, this periodogram did not reveal anything distinguishable from the noise (Fig. 2, top frame), despite of a very large amplitude of the RV variation. After that, we applied more advanced multiharmonic periodograms (Schwarzenberg-Czerny 1996; Baluev 2009b) to the data. In these periodograms the signal is modelled as a sum of a few of first Fourier terms, which is more adapted to non-sinusoidal variations. A disappointing thing is that even these periodograms do not help (Fig. 2, second and third plots). Even the model with 10 Fourier harmonics is in fact useless here: the periodogram still looks like a wide-band noise without a hint of any clear isolated peak. Although we can see some moderate peak at the true period $P = 111$ d, there are a lot of other peaks at different periods. We would be unable to identify the correct peak until we know the true period. At last, we proceed to the Keplerian periodograms (Fig. 2, fourth and fifth plots). We considered to values of e_{\max} here, 0.6 and 0.9. While the

¹ It appeared that Wittenmyer et al. (2009) released an improved version of the HET RV data for HD 80606. But we discovered this already after our simulations were done, and we decided not to re-run them, as we pursue only demonstration goals here.

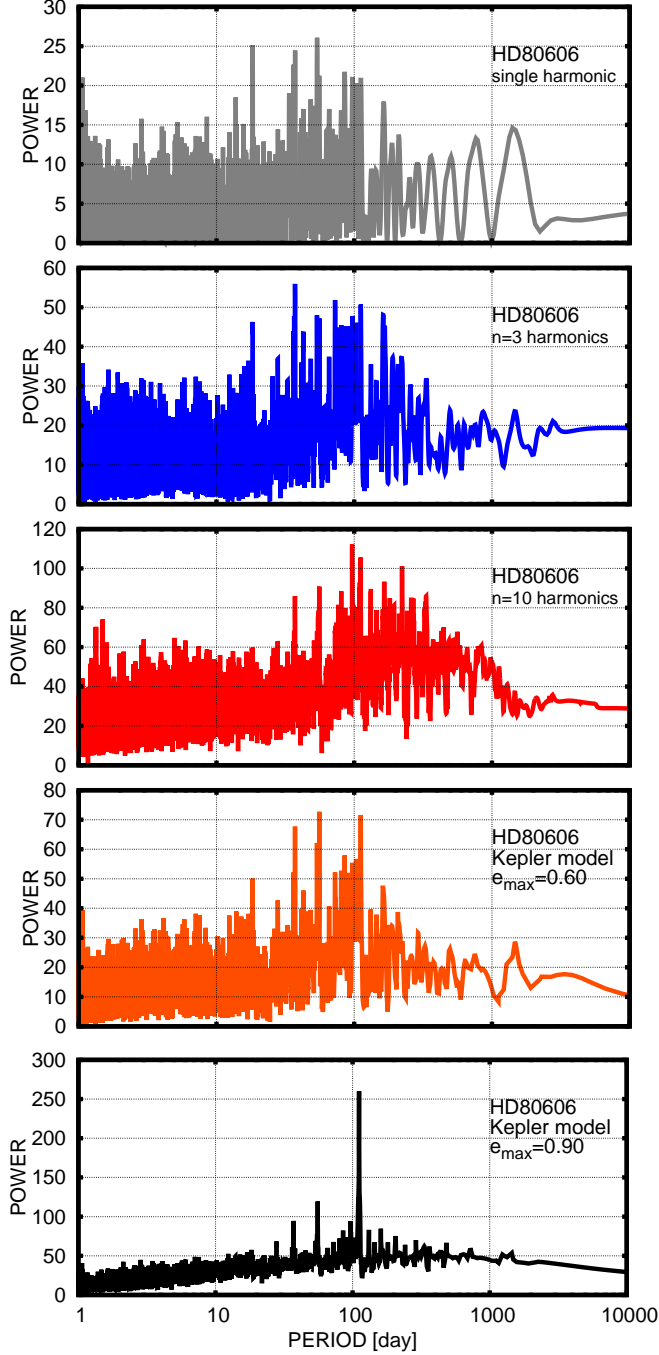


Figure 2. Periodograms of the Doppler RV data for the star HD80606. From top to bottom: the single-harmonic periodogram, two multiharmonic periodograms of different order, two Keplerian periodograms with different maximum eccentricity.

periodogram for $e_{\max} = 0.6$ still remains rather unimpressive, the periodogram for $e_{\max} = 0.9$ contains a clear and undoubtful peak at the correct period value of 111 d.

This test case emphasizes the importance of careful processing of large eccentricities. We could not detect the planet until we reach the values of e as large as 0.9, and this parameteric domain we must carefully sampled in order to not miss any maxima of the likelihood function. This is what our computation algorithm is aimed on. Unfortunately, dealing

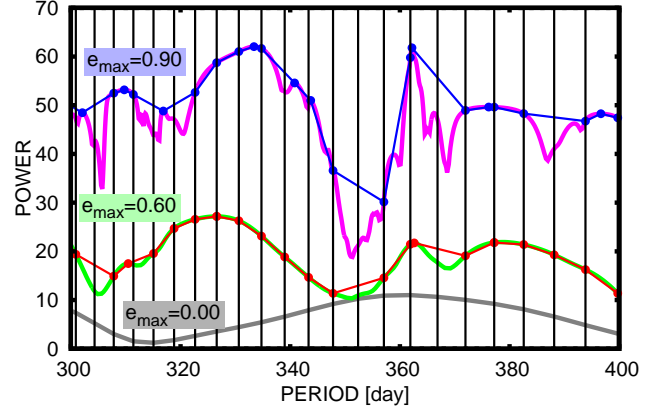


Figure 3. The fine structure of the single-harmonic and Keplerian periodogram of the HD80606 RV data. We plot short magnified pieces of two Keplerian periodograms from Fig. 2 (points connected with line segments), and compare them with their high-resolution versions (smooth lines). The grid of vertical lines maps to the frequency grid adopted in Fig. 2.

with large eccentricities $e > 0.6$ requires quickly increasing computation resources. However, we treat this as a necessary sacrifice.

Now, let us consider how our algorithm works in more fine details. In Fig. 3 we plot a zoomed graph of the single-harmonic periodogram and the Keplerian periodograms for $e_{\max} = 0.6$ and $e_{\max} = 0.9$. Note that to reduce the size of the output file, our Keplerian periodogram algorithm also allows to choose an arbitrarily large effective (or userspace) frequency step, keeping internally the fine grid necessary to adequately cover all likelihood maxima. The user can set the frequency step even much larger than the typical width of the periodogram peaks. The Keplerian periodogram is internally maximized in each of the large chunks of the user-specified frequency grid, and only the maxima attained within the chunks are saved in the file. We call this a “sparse” algorithm of periodogram evaluation. Two Keplerian periodograms shown in Fig. 3 are plotted for two user-specified resolutions: a version with very fine resolution (smooth curves) and a version with low resolution (points connected by line segments). We also overplot the frequency grid inferred by the latter (low) resolution.

First, we can see that $e_{\max} = 0.9$ indeed generates much finer structures in the periodogram than $e_{\max} = 0.6$, as follows from the discussion in Sect. 5. If we would try to obtain the $e_{\max} = 0.9$ periodogram fully resolved and in the entire frequency segment of Fig. 2, we would deal with a huge output file and an increased computation time. Although we sacrificed the density of frequency grid, we did not loose any important information about the periodogram peaks. Each point of the output grid corresponds to either a local maxima inside a grid chunk, or to a boundary of the chunk, depending on which periodogram value appears larger. Thus, armed with this algorithm, we never loose any of the major peaks on the Keplerian periodogram. In the ultimate case, we may even request to cover the entire frequency range by a single step. Then the algorithm will just compute the absolute maximum of the Keplerian periodogram attained over this frequency segment. The default frequency resolu-

tion adopted in PlanetPack is such that approximately a single output value is supplied per each local maximum of the plain single-frequency periodogram. This means that in Fig. 3 we would have only a single such point, corresponding to the maximum found within this range.

7 INVESTIGATING THE DETECTION EFFICIENCY OF THE KEPLERIAN PERIODOGRAM

In this section we undertake an attempt to quantitatively characterize the detection efficiency of the Keplerian periodogram. This is a rather rough investigation. We do not try to deal with probabilistic efficiency characteristics like e.g. the probability of planet detection for given Keplerian parameters. Instead, we only operate with the expected detection thresholds, roughly corresponding to some intermediary (e.g. median) detection probability. Also, we neglect the aliasing effects (or spectral leakage), which are caused by an interference between the periodic gaps in the RV data and the periods of the RV variation. We adopt the UPC (Uniform Phase Coverage) approximation noted above, in which the averages over the time series are approximated by continuous integrals. Our goal here is to characterize the detection efficiency of the Keplerian periodogram from the most general point of view, instead of e.g. binding to the data with particular characteristics.

The power of a sinusoidal signal, with a given amplitude of K , is determined trivially through the integration along its single period:

$$\begin{aligned} \mathcal{P}_{\sin}(K) &= \int_0^P K^2 \cos^2 \left(2\pi \frac{t}{P} + \lambda_0 \right) \frac{dt}{P} = \\ &= \int_0^{2\pi} K^2 \cos^2 \lambda \frac{d\lambda}{2\pi} = \frac{K^2}{2}. \end{aligned} \quad (29)$$

The power of the Keplerian signal (1) can be expressed in a similar way as

$$\begin{aligned} \mathcal{P}_{\text{Kep}}(K, \beta, \omega) &= \int_0^P \mu^2(t, \theta) \frac{dt}{P} = K^2 \int_0^{2\pi} g^2(\lambda, \nu) \frac{d\lambda}{2\pi} = \\ &= \frac{K^2}{2} \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2 (1 - \beta^2 \cos 2\omega). \end{aligned} \quad (30)$$

This result can be also found in the attached MAPLE worksheet.

In what follows below we largely rely on the assumption that whenever the RV data carry a signal indeed, the periodograms maxima should be approximately proportional to the power of this signal, computed according to (29) or (30).

This property is easier to demonstrate for the simplest χ^2 periodograms designated above as $z(f)$ this property. First, we “denoise” this χ^2 function (2) by replacing the measurements x by the actual variation (containing the real signal), and after that apply the UPC approximation:

$$\chi_{\mathcal{H}, \kappa}^2 = \langle (x - \mu_{\mathcal{H}, \kappa})^2 \rangle \simeq \langle (\hat{\mu}_{\mathcal{H}} - \mu_{\mathcal{H}, \kappa})^2 \rangle \simeq$$

$$\simeq \langle 1 \rangle \int_{t_{\min}}^{t_{\max}} (\hat{\mu}_{\mathcal{H}} - \mu_{\mathcal{H}, \kappa})^2 \frac{dt}{T}, \quad T = t_{\max} - t_{\min}. \quad (31)$$

Here the notation $\hat{\mu}_{\mathcal{H}}$ stands for the actually present cumulative variation, having the same functional shape as $\mu_{\mathcal{H}}$ (i.e. underlying variation + signal) with some adopted “true” values of the parameters.

To derive the maximum of the relevant periodogram we need to minimize the functions $\chi_{\mathcal{H}, \kappa}^2$ by varying the arguments $\theta_{\mathcal{H}, \kappa}$ in the models $\mu_{\mathcal{H}, \kappa}$, but keeping the analogous parameters in $\hat{\mu}_{\mathcal{H}}$ fixed at their prescribed values. Clearly, in the approximation (31), the global minimum for $\chi_{\mathcal{H}, \kappa}^2$ is zero, achieved when the parameters in $\mu_{\mathcal{H}}$ coincide with those in $\hat{\mu}_{\mathcal{H}}$. To handle $\chi_{\mathcal{H}}^2$, we can apply a linearization of the base model $\mu_{\mathcal{H}}$, if this model is not already linear in itself:

$$\chi_{\mathcal{H}}^2 \simeq \langle 1 \rangle \int_{t_{\min}}^{t_{\max}} (\hat{\mu}_{\mathcal{H}} - \mu_{\mathcal{H}} + \hat{\mu})^2 \frac{dt}{T}. \quad (32)$$

Now, if the terms of the type

$$\int_{t_{\min}}^{t_{\max}} \mu(t, \theta) \mu_{\mathcal{H}}(t, \theta_{\mathcal{H}}) dt \quad (33)$$

can be neglected then the integral in (32) can be split in two independent nonnegative terms as

$$\chi_{\mathcal{H}}^2 \simeq \langle 1 \rangle \left[\int_{t_{\min}}^{t_{\max}} (\hat{\mu}_{\mathcal{H}} - \mu_{\mathcal{H}})^2 \frac{dt}{T} + \int_{t_{\min}}^{t_{\max}} \hat{\mu}^2 \frac{dt}{T} \right], \quad (34)$$

and then the minimum of $\chi_{\mathcal{H}}^2$ is achieved for $\theta_{\mathcal{H}} = \hat{\theta}_{\mathcal{H}}$, when $\mu_{\mathcal{H}}$ coincides with $\hat{\mu}_{\mathcal{H}}$. In this case we can easily obtain the desired result:

$$z = \frac{1}{2} (\min \chi_{\mathcal{H}}^2 - \min \chi_{\mathcal{K}}^2) \simeq \frac{\langle 1 \rangle}{2} \int_{t_{\min}}^{t_{\max}} \hat{\mu}^2 \frac{dt}{T} \simeq \frac{\langle 1 \rangle}{2} \int_0^P \hat{\mu}^2 \frac{dt}{P}. \quad (35)$$

As the time range T is usually large, we have replaced in the last formula the integration along the entire segment $[t_{\min}, t_{\max}]$ by an integral along only a single period of the signal. The last integral represents the power of the signal, like (29) or (30).

The condition that (33) should be negligible is an orthogonality condition between the signal and the base models. In our assumptions, and for a sinusoidal or a Keplerian signal, it is usually satisfied. For example, frequently $\mu_{\mathcal{H}}$ contains only a constant, and in this case we only need the signal to be properly centred, satisfying $\int_0^P \mu dt = 0$. For a linear or quadratic underlying variation, the orthogonality condition is also approximately fulfilled, if the time range covers many periods of the signal (i.e. this period is short in comparison with T), and we neglect aliasing effects. Similarly, whenever any previously detected planets persist in $\mu_{\mathcal{H}}$, the orthogonality condition is approximately satisfied, unless periods of some of these planets are close to the period of the signal, which is an impractical case, or they settle an interference with the signal via the aliasing mechanism, which we neglect here.

When the existing signal is Keplerian, and with the use of the Keplerian model, we may accumulate the full Kep-

lerian power (30) in the observed periodogram maximum. Using (35) we may approximate this maximum as

$$z_{\text{Kep}} \simeq \frac{\langle 1 \rangle}{2} P_{\text{Kep}}(K, \beta, \omega). \quad (36)$$

Note that this approximation is valid only if the signal eccentricity does not exceed the maximum one allowed in the computation of the Keplerian periodogram.

But if we use a sinusoidal model to detect a Keplerian signal we would deal with multiple periodogram peaks corresponding to various Fourier subharmonics of (1). The height of each such peak would be proportional to the power of the relevant sinusoidal subharmonic:

$$z_{\text{sin}} \simeq \frac{\langle 1 \rangle}{2} P_{\text{sin}}(K A_{\text{max}}(\beta, \omega)), \quad (37)$$

where A_{max} is the maximum amplitude among the Fourier subharmonics for the normalized Keplerian function $g(\lambda, \nu)$.²

Thus, from (36) and (37) we can approximate the maxima ratio for the Keplerian and a sinusoidal (e.g. Lomb-Scargle) periodograms like:

$$\frac{z_{\text{Kep}}}{z_{\text{sin}}} \simeq \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2 \frac{(1 - \beta^2 \cos 2\omega)}{A_{\text{max}}^2(\beta, \omega)}. \quad (38)$$

This ratio does not depend on K .

To compute the quantity A_{max} in (37) and (38), we must use the Fourier coefficient of the Keplerian RV function, that become pretty easy to compute using the formulae of the mentioned above integrals of $r^n \cos kv$ in (Kholoshevnikov & Titov 2007). Designating the cosine Fourier coefficients as c_k , and the sine coefficients as s_k , we may obtain:

$$c_k = 2 \frac{1 - e^2}{e} J_k(ke) \cos \omega, \quad s_k = 2\eta J'_k(ke) \sin \omega, \quad (39)$$

where J_k are Bessel functions. Finally, the desired quantity A_{max} can be computed as:

$$A_{\text{max}}^2 = \max_{k \geq 1} (c_k^2 + s_k^2). \quad (40)$$

From the other side, based on the FAP formulae above, we can compute the approximate detection thresholds for the relevant periodograms, given some small critical FAP = α :

$$z_{\text{sin}}^{\text{thr}}(\alpha, W), \quad z_{\text{Kep}}^{\text{thr}}(\alpha, W, \beta_{\text{max}}). \quad (41)$$

The maximum peak observed in the Keplerian periodogram is larger than the maximum of the sinusoidal periodogram, because the Keplerian periodogram is able to accumulate the full power of the signal. But from the other side, the noise level in the Keplerian periodogram is higher than in the sinusoidal one, because the Keplerian model has more free parameters, and also because the expected number of the noisy peaks in the Keplerian likelihood function grows quickly when β_{max} increases. The main question is: which tendency wins, depending on the signal's β and ω ?

So far we did not put any constraints on the signal amplitude K . Now, let us assume that the signal amplitude

K is such that for the sinusoidal periodogram we have a boundary detection:

$$z_{\text{sin}}/z_{\text{sin}}^{\text{thr}} = 1. \quad (42)$$

With this prerequisite, for the Keplerian periodogram of the same signal we can easily compute the analogous detection efficiency ratio:

$$\frac{z_{\text{Kep}}}{z_{\text{Kep}}^{\text{thr}}} = \frac{z_{\text{sin}}^{\text{thr}}}{z_{\text{Kep}}^{\text{thr}}} \frac{z_{\text{Kep}}}{z_{\text{sin}}}. \quad (43)$$

In fact, the quantity (43) represents the relative detection efficiency of the Keplerian periodogram comparatively to the ones utilizing a sinusoidal signal model. Selecting some reasonable values for α , W , and β_{max} , we can numerically compute the threshold ratio in (43). The remaining ratio $z_{\text{Kep}}/z_{\text{sin}}$ is approximated in (38) through a function of β and ω . Therefore, the detection efficiency (43) can be viewed as a function of a location in the (β, ω) plane. Values of (43) exceeding unit indicate the expected advantage of the Keplerian periodogram, while values below unit indicate the advantage of the sinusoidal model. The square root of this function measures the relative efficiency in terms of the signal amplitude K (rather than K^2 , which is a less intuitive quantity).

This relative detection efficiency is plotted in Fig. 4 as a function of the planetary eccentric parameters $e \cos \omega$ and $e \sin \omega$. The two plots assume two values of e_{max} , respectively 0.6 and 0.9. The FAP detection threshold was set to $\alpha = 0.01$ and the frequency range to $W = 5000$ (other reasonable values did not lead to any remarkable changes in the plots). First, we can see from these plots that they do not reveal any dramatic difference in the detection efficiency. In the case $e_{\text{max}} = 0.6$ the relative efficiency always stays close to unit. For $e_{\text{max}} = 0.9$, the minimum efficiency of the Keplerian periodogram does not fall below 0.8 of that of the Lomb-Scargle one. As expected, this minimum corresponds to planets with zero eccentricities, for which the use of the Keplerian model is unnecessary. The maximum relative efficiency reaches a moderate value of 1.7. corresponds to the points $\omega = 0$ or $\omega = \pi$ located on the boundary $e = e_{\text{max}}$. In fact, for larger e its relative efficiency should grow further, but in the plots we cut out these regions, because the formula (30) becomes invalid there and we are unable to correctly compute the efficiency function there.

One might draw a conclusion from Fig. 4 that the Keplerian periodogram can advance the detection power of highly eccentric planets only pretty moderately, although for the unfavoured almost circular orbits it does not introduce a significant degradation as well. However, these results were obtained using a pretty rough and simplified treatment, and such a conclusion would not be entirely objective. The practical advantage of the Keplerian periodogram for large eccentricities does not relies on only the increased height of the periodogram peaks. In Fig. 2, top panel, we can see that the most important issue with the sinusoidal periodogram is that it is unable to reveal a well isolated period of the eccentric planet. This periodogram still looks like a wide-band noise. Formally, its noise level is larger than we would expect from the data with no signal at all: e.g. the estimation (11) yields $\text{FAP} \sim 10^{-7}$. However, this information remains rather useless, because we are unable to locate a clear period and even to suspect that such a period exists.

² Note that for large eccentricities, the primary subharmonic is not necessarily the one with the maximum amplitude.

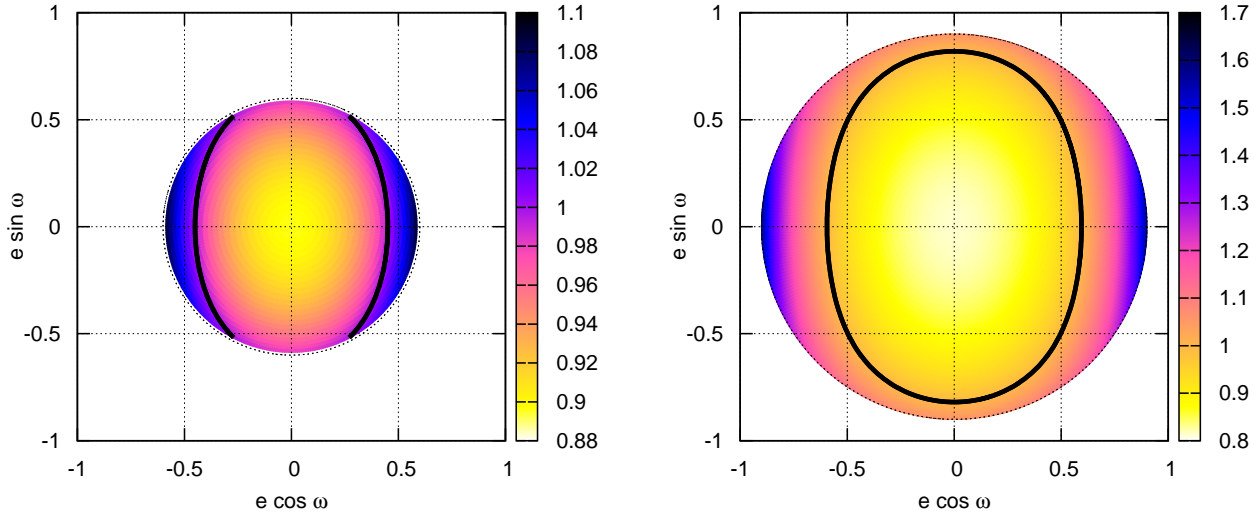


Figure 4. Relative detection efficiency of the Keplerian periodogram comparatively to the Lomb-Scargle one, plotted as a color-mapped function of the planet orbital eccentricity e and the pericenter argument ω . The circular outline of the plots labels the value of e_{\max} adopted in the Keplerian periodogram ($e_{\max} = 0.6$ in the left panel and 0.9 in the right one). The oval thick line labels the points of equal detection efficiency (unit relative efficiency). In the both plots we adopted the FAP threshold of $\alpha = 0.01$, and the frequency bandwidth of $W = 5000$. See text for more comments and definitions.

The Keplerian periodogram solves this task gracefully, and not just because the height of its peaks is plainly larger. It allows to locate a clearly isolated Keplerian period, indicating that the data indeed contain a periodic signal. Moreover, from the results by O’Toole et al. (2009b) it follows that the Keplerian periodogram may become pretty useful even for moderate eccentricities of $e \sim 0.6$. In this example it helped to disentangle multiple Keplerian signals from each other, which the sinusoidal periodograms were unable to achieve.

In fact, the main purpose of Fig. 4 here is to demonstrate that when we are dealing with planets having *small* eccentricities, for which our efficiency indicator is more adequate, the Keplerian periodogram still does not cause a significant degradation in the detection power. This means that the Keplerian periodogram does not impose any difficult trade-off between the detection of only low-eccentricity or only high-eccentricity planets. This property becomes important for systems that contain planets on orbits with low and high eccentricities simultaneously: we could observe the hints of all such planets in the same Keplerian periodogram. The sinusoidal periodogram would, at the best, reveal only the periods of the low-eccentricity planets, presenting the other planets as a noisy mesh of peaks.

8 PRACTICAL VALIDITY OF THE KEPLERIAN SIGNIFICANCE THRESHOLDS

The analytic FAP estimates from Sect. 4 were derived for pretty simplified and apparently restrictive conditions. They are summarized here:

(i) The simplified chi-square objective function (2) is adopted, implying that the periodogram are based on the chi-square statistic z . This infers an assumption of fixed and a priori known noise variances. In practice we usually do not know the noise level well, implying that we have to use some parametrized noise model. This would make use of the likelihood function (6) and likelihood-ratio test statistic Z or \tilde{Z} , involving e.g. an additive jitter model. This is different from what we adopted in Sect. 4.

(ii) The base model $\mu_{\mathcal{H}}$ is assumed strictly linear. In practice this is true only if we have not detected even a single planet yet, implying that $\mu_{\mathcal{H}}$ only involves a free RV offset, constant in time. But after the detection of the first planet, the model $\mu_{\mathcal{H}}$ becomes non-linear. Even if this planet had zero eccentricity and its RV signal was sinusoidal, at least the period of the sinusoid would be a non-linear parameter.

(iii) To obtain entirely analytic FAP estimations in a closed form, the approximation of the “uniform phase coverage” (UPC) was used extensively. This means that various summations over the time series were approximated by continuous integrals over a single period of μ . In practice this approximation may be bad for uneven time series, in particular when the signal period is in a commensurability with some periodic leaks in the RV data. Also, this approximation may fail when the function to be integrated contains narrow spikes or peaks that may fall in the gaps between discrete observations. Such effect can emerge for large eccentricities.

Fortunately, a violation of these assumptions does not necessarily corrupt the accuracy of the final FAP estimations very much. Here is the justification:

(i) Various statistical properties of the chi-square test are

often applicable to the likelihood-ratio one in an approximate (asymptotic) sense, under an extra condition $N \rightarrow \infty$. This issue is considered in (Baluev 2014b), where it is also demonstrated that the effect of a non-trivial noise model is similar in its nature to the effect of non-linearity in the RV curve models, $\mu_{\mathcal{H}}$ in our case. Often this model can be linearized in a small ($\sim 1/\sqrt{N}$) vicinity of the best fit parameters, again making the results of Sect. 4 applicable approximately for large N . More accurately, we should satisfy the asymptotic condition of the type $Z, \tilde{Z} \ll \mathcal{O}(N)$ (Baluev 2009a). The results of Sect. 4 have asymptotic nature themselves, requiring $z \rightarrow \infty$. Therefore, we must have \tilde{Z} large enough to have the formulae from Sect. 4 useful, but \tilde{Z} must not be too much large, in order to avoid the non-linearity effects. In practice it is enough to have good accuracy in a rather limited range $\text{FAP} \in [10^{-1} - 10^{-3}]$: for small \tilde{Z} levels the FAP appears too large anyway, implying an insignificant signal, and for very large \tilde{Z} it is anyway safely small (not a big deal, how much small).

(ii) As explained in (Baluev 2013c), the spectral leakage has a negligible effect on the accuracy of the UPC approach, because the UPC approximation is invalidated only in a few of very narrow frequency segments, associated to the peaks of the spectral window function, while the necessary integrals usually involve a wide frequency range. On contrary, the issue with failing UPC at large eccentricities may be potentially important, although this effect is unrelated to the spectral leakage. When UPC is failed, the coefficients X and Y in (21) should be computed using direct time-series summations (Baluev 2013c, sect. 4.2). We have little to simplify or detail here: we should just substitute the parametric derivatives of the Keplerian RV model (1) and of the adopted $\mu_{\mathcal{H}}$ in that formulae. Although more general and accurate, this method should be avoided whenever possible, because it is computationally expensive (its complexity is comparable to a single evaluation of the Keplerian periodogram), and the result is useful only for a particular time series.

As previous simulations revealed (Baluev & Beaugé 2014; Baluev 2014b), the effect of non-linear $\mu_{\mathcal{H}}$ as well as the effect of the non-trivial noise model may increase FAP in comparison with what expected from the linear case. This may even break the inequality in (9). However, the inaccuracy of the UPC approximation likely has an opposite effect, leading to an overestimated FAP (Baluev 2013c). We do not expect it to break the inequality of (9). Whether or not these effects are significant at all, depends on the particular case. In this section our goal is to verify the applicability of the FAP estimations under various conditions typical for exoplanetary Doppler surveys.

In the first example we consider the 51 Pegasi RV data acquired by ELODIE spectrograph (Naef et al. 2004). Their time distribution is more or less typical for ground-based surveys, involving seasonal gaps and diurnal regularity. Their number is relatively large, $N = 153$, and the base model is rather simple, involving only a single planet on almost circular orbit (implying only a single sinusoidal RV term). Such simple model is very well linearizable. Concerning the RV noise in this data, we approximate it with the usual model involving an additive jitter (Baluev 2009a; Wright 2005). In another study we have already verified that it does not generate any significant additional non-linearity

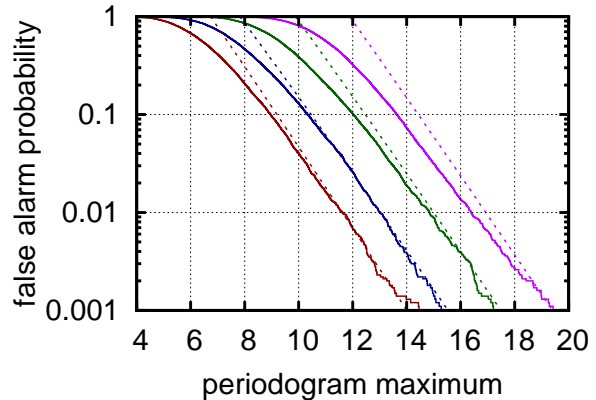


Figure 5. Comparison of the simulated FAPs of the Keplerian periodogram with their analytic estimations: the case of the 51 Peg ELODIE RV data. The graphs show the simulated FAPs (solid curves) with their theoretic approximations (broken lines) for $e_{\max} = 0, 0.3, 0.6$, and 0.8 (from left to right). All the cases have $f_{\max} = 0.1 \text{ d}^{-1}$.

effects with these particular data (Baluev 2014b). When dealing with a Keplerian periodogram of these data, we may face only the non-linearity of the probe Keplerian signal (1). This makes the task very close to the idealized conditions of Sect. 4.

Monte Carlo simulations presented in Fig. 5 confirm this. We can see that the simulated FAP curves are in a good agreement with the analytic formulae (21,24). Monte Carlo simulation of the Keplerian periodogram is an extremely heavy task in view of the computational resources, so we had to limit the periodogram to a rather narrow frequency range with $f_{\max} = 0.1 \text{ d}^{-1}$, although the more practical and typically adopted value is $f_{\max} = 1 \text{ d}^{-1}$. We sacrificed the frequency range in order to allow large enough values of the eccentricity limit e_{\max} , because the specific of the Keplerian periodogram is in the variable eccentricity. In fact, this further highlights the usefulness of the analytic FAP estimation for the Keplerian periodogram: for more practical values of f_{\max} and e_{\max} it is still possible to compute a single Keplerian periodogram, but it would be just infeasible to simulate its FAP levels by Monte Carlo.

Another simulation refers to the RV data of HD 80606, already considered in Sect. 6. In this case we adopted more complicated base model, including the high-eccentricity planet b , independent offsets for the RV subsets coming from different observatories, and an additional sinusoidal annual variation in the HET RV data from (Wittenmyer et al. 2007). The latter variation was partly motivated by the presence of a similar annual variation in some HET RV data for HD 74156 (Baluev 2009a; Meschiari et al. 2011). Although that HD 74156 data were acquired by another team Bean et al. (2008), and we did not actually detect a significant annual variation in the HD 80606 data, we added this annual term to the RV model to make it more complicated. Finally, we replaced the additive RV noise model with the regularized one from (Baluev 2014b) to suppress possible interfering non-linearity generated by the RV noise rather than RV curve.

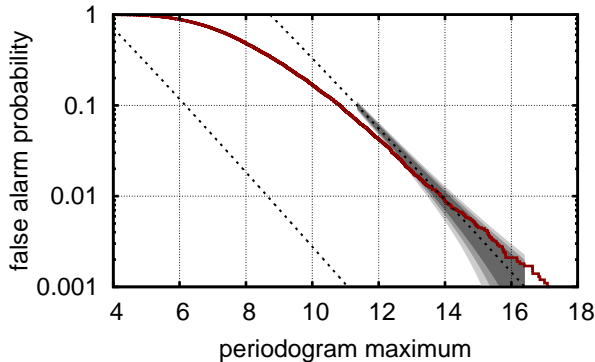


Figure 6. Comparison of the simulated FAP of the Keplerian periodogram with the analytic estimations: the case of the HD 80606 RV data. The simulated FAP for $e_{\max} = 0.9$ and $f_{\max} = 0.01 \text{ d}^{-1}$ is shown as a solid curve. The first broken line (in the left part) corresponds to the analytic FAP for the Lomb-Scargle periodogram and is shown here for comparison. The second broken curve shows the analytic approximation to the Keplerian FAP. The gray-filled areas around this line correspond to the 1, 2, 3-sigma Monte Carlo uncertainty ranges (see text for details).

The simulation results for this case are presented in Fig. 6. The frequency limit here was reduced to $f_{\max} = 0.01 \text{ d}^{-1}$, to allow the higher eccentricity limit of $e_{\max} = 0.9$. We can see that for large periodogram peaks, or equivalently small FAP levels, the simulated FAP curve passes slightly above the theoretic prediction, formally breaking the inequality (9). This may indicate the presence of some modest non-linearity in the base RV model. In this case we may note that the impact of non-linearity is smaller than we might expect. In the base model we have a $e = 0.93$ Keplerian signal of the known planet, which is an extremely non-linear function from the first view. Nonetheless, even this modest FAP increase is consistent with the Monte Carlo uncertainties (shown in Fig. 6 as gray-filled ranges near the theoretically predicted curve). The latter uncertainties were constructed by means of a tail-weighted Kolmogorov-Smirnov test (Chicheportiche & Bouchaud 2012; Baluev 2014b).

9 CONCLUSIONS AND DISCUSSION

Although a lot of period finding methods are currently available for an astronomer (Graham et al. 2013), it is important to avoid an unjustified use of a method in the tasks in which it was not supposed to properly work. For example, the old approach (Horne & Baliunas 1986) approximates the periodogram FAP levels on the basis of the assumption of independent periodogram readouts. For the Lomb-Scargle periodogram this approach approximates the FAP with a formula:

$$\text{FAP} \approx 1 - (1 - e^{-z})^{N_{\text{ind}}}, \quad (44)$$

where N_{ind} is the effective number of independent periodogram readouts (or “independent frequencies”). The formula (44) is formally valid only under pretty strict conditions, namely (i) the time series should be evenly spaced, and (ii) the periodogram of these data is computed only on a discrete and rather sparse set of the fundamental frequencies

(no oversampling). As these conditions are rarely fulfilled in the astronomical practice, Horne & Baliunas (1986) suggested to use (44) just as an extrapolating formula, in which the quantity N_{ind} is treated as a free parameter fitted by Monte Carlo simulations. We do not criticize here this idea itself, because it was rather reasonable and in fact the most usable approach among those available in 1986. However, today we should remember that in the Horne & Baliunas (1986) treatment the formula (44) have lost its theoretical basis and serves as just a parametric FAP fitting formula. Its accuracy is not guaranteed in the individual practical cases.

Moreover, numerous later works have already revealed multiple theoretic as well as practical weaknesses of the formula (44), see e.g. (Koen 1990; Frescura et al. 2008; Baluev 2008; Süveges 2014). For example, for large z the formula (44) yields the approximation $\text{FAP} \sim N_{\text{ind}} e^{-z}$, which is by the factor of \sqrt{z} different from the correct asymptotic FAP behaviour, given by (11). Although this \sqrt{z} factor could be compensated by selecting a larger N_{ind} , to achieve this we should significantly increase the number of Monte Carlo trials to cover the smaller FAP levels reliably. However, in this case the use of the formula (44) becomes just senseless, because in this case we could just estimate the FAP from these simulations directly.

The modern FAP estimation that were originally introduced in (Baluev 2008) for the Lomb-Scargle periodogram, and now extended to the Keplerian signal model, are free from the issues of the Horne & Baliunas (1986) approach. Our technique has a strict and general theoretic basis of the Rice method, while its final formulae are usually very simple and do not require any Monte Carlo simulations at all. Moreover, this entirely analytic approximation in practice usually appears more accurate than the approximation (44), even if we fit N_{ind} in the latter by simulations. For example, Hartman et al. (2014) says that with the Horne & Baliunas (1986) method “the resulting false alarm probability may be inaccurate by as much as a factor of ~ 10 ”, and this seems to be even a rather optimistic assessment. In view of this, it appears rather strange and unexplainable that an almost 30-years age method, which is already known for its practical deficiency, is still so frequently used in the astronomical practice (e.g. Burt et al. 2014; Brothwell et al. 2014; Kelly et al. 2014; Hartman et al. 2014; Nucita et al. 2014; Chen et al. 2014; Lin et al. 2014; Romano et al. 2014; Buccino et al. 2014; Brucalassi et al. 2014, among publications of only the year of this writing).

It might be indeed a bit scaring to blindly rely on an entirely analytic formula like (11) without any Monte Carlo calibration. However, instead of fitting the archaic formula (44), which is already known to be not accurate, it might be considerably more informative to just give a comparison of the simulated periodogram distribution with (11) and (44). Alternatively, it might be recommended to fit the FAP with some more suitable approximation, e.g. with the two-parametric formula

$$\text{FAP} \approx A e^{-z} z^p, \quad (45)$$

with A and p determined by Monte Carlo. This formula represents a general form for the primary FAP term that is usually obtained after applying the Rice method to various periodograms. Also, Süveges (2014) suggested to fit the FAP with the three-parametric generalized extreme-value distribution.

bution. As well as the Rice method, this new approach has a general mathematical basis too, although it requires Monte Carlo, and the limits of its applicability and accuracy look different.

In view of the increased complexity of the signal model used in the Keplerian periodogram, its practical application is a computationally heavy task. In this concern, analytic methods that allow to avoid Monte Carlo simulations become especially precious. The main result of this paper is the analytic FAP approximation for the Keplerian periodogram given by the formulae (21) and (24) of Sect. 4. As we can see, these formulae are in fact elementary and should be definitely helpful in practical computations. These FAP estimations, together with the computation algorithm of Sect. 5, are now implemented in PlanetPack, a public open-source software for Doppler time series analysis (Baluev 2013a). Note that some computation algorithm for the Keplerian periodogram was included in PlanetPack 1.6 and onwards, but that was only a preliminary experimental version without any FAP estimations. The new algorithms will be soon released with the forthcoming PlanetPack 2.0.

ACKNOWLEDGEMENTS

This work was supported by the President grant for young scientists (MK-733.2014.2), by the Russian Foundation for Basic Research (projects No. 12-02-31119 mol_a and 14-02-92615 KO_a), and by the programme of the Presidium of Russian Academy of Sciences “Non-stationary phenomena in the objects of the Universe”. The PlanetPack C++ code includes a snapshot of the C library for stellar limb-darkening models from (Abubekrov & Gostev 2013), by the kind permission of the authors. I would like to express my gratitude to the anonymous referee for providing insightful comments.

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